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m-ARY Partitions

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m-ARY PARTITIONS

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ABSTRACT

We study the number of ways in which $n \geq 0$ be written as the sum of powers of $m \geq 2$. After briefly discussing historical results and examples, we prove recurrence relations, exact formulae, bounds, and asymptotic formulae. The proofs are based on elementary reasoning using induction and the mean value theorem.

DEDICATION

To my dear parents

Soumitra and *Indira*

To my sweet sister

Soumita

To my lovely grandmother

Shefali

In honor of

The grammarian *Pāṇini*

And the mathematician *Āryabhaṭa*

Who were among the wisest in Āryāvarta

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Chapter 1

Introduction

Let $m \geq 2$. Consider $\{P_m(n)\}_{n=0}^{\infty}$, or "$m$-ary partitions", where $P_m(n)$ is the number of solutions of

$$n = i_1 + i_2 m + i_3 m^2 + \dots$$

in non-negative integers $i_k, k \geq 1$. Stated in a different way, $P_m(n)$ is the number of ways of writing n as a sum of powers of m that are greater than or equal to 1. For $m = 2$, they are known as binary partitions. For example, $P_2(6) = 6$ since

$$\begin{aligned} 6 &= 1+1+1+1+1+1 \\ &= 1+1+1+1+2 \\ &= 1+1+2+2 \\ &= 1+1+4 \\ &= 2+2+2 \\ &= 2+4 \end{aligned}$$

Here we present the reader with a short section on the history of m-ary partitions; this information was gathered primarily from the work of Reznick [4] and from that of Pennington [30]. Euler [24] (Latin) computed $P_2(n)$ for $n \leq 37$. He knew that

$$P_2(n) = P_2(n-2) + P_2\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \tag{1.1}$$

This easily gives the following recurrence relation

$$P_2(n) = P_2(0) + \cdots + P_2\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \tag{1.2}$$

which is present in Tanturri [3]. Proofs of these formulas are present in chapter 2. Reznick remarks that Tanturri wrote a series of papers on binary partitions during World War I; he used the now obsolete symbolic notation of Peano to express his formulas; see [1] (Italian), [2] (Italian), [3] (Italian). Knuth [6] studied the growth of $P_2(n)$ and gave some recurrences for $P_2(n)$ in 1966; he remarks that 1.1 immediately gives

$$(P_2(4n))^2 = P_2(4n-2)P_2(4n+2) + (P_2(2n))^2$$

In 1969, Churchhouse [28] studied $P_2(n) \pmod{2^r}$. With $\nu_2(n)$ being the largest power of 2 dividing n, $2 \mid P_2(n)$ for $n \geq 2$, $4 \mid P_2(n)$ if and only if $\nu_2(n)$ or $\nu_2(n-1)$ is positive and even, and $8 \nmid P_n(n)$ for all n. Churchhouse conjectured that if n is even, then

$$\nu_2(P_2(4n) - P_2(n)) = \left\lfloor \frac{3\nu_2(n) + 4}{2} \right\rfloor$$

This conjecture was proved by Rødseth [27], Gupta ([21], [19], [18]), and generalized by Hirschhorn and Loxton [26] in 1976. The result was also generalized to m-ary partitions for $m > 2$ by Rødseth, Gupta [20], Andrews [15], and Gupta and Pleasants [22]. Churchhouse iterated 1.2 to express $P_2(2^r n)$ in terms of $P_2(i)$ for $0 \leq i \leq n$. This idea has been generalized to m-ary partitions for $m > 2$. We use this idea in chapter 3 to independently obtain an explicit formula for $P_m(n)$. Restricted m-ary partitions ($i_k = 0$ for $k \geq t$) have been studied by Gupta and Pleasants, Dirdal ([13], [12]); a summary of this is present in [16]. Reznick [4] is an excellent paper on the d^{th} binary partitions: $m = 2$, $0 \leq i_k \leq d-1$ for $k \geq 1$.

In 1940, Mahler [23] obtained solutions of the functional equation

$$\frac{f(z+\omega) - f(z)}{\omega} = f(qz)$$

and as an application, he proved that

$$P_m(mn) = e^{\mathcal{O}(1)} \sum_{i=0}^{\infty} \frac{m^{-\frac{1}{2}i(i-1)} n^i}{i!}$$

which leads to

$$\begin{aligned}\log(P_m(mn)) = \frac{1}{2\log(m)} \left(\log\left(\frac{n}{\log(n)}\right)\right)^2 + \left(\frac{1}{2} + \frac{1}{\log(m)} + \frac{\log\log(m)}{\log(m)}\right)\log(n) \\ - \left(1 + \frac{\log\log(m)}{\log(m)}\right)\log\log(n) + \mathcal{O}(1)\end{aligned}$$

In 1948, De Bruijn [5] showed that the $\mathcal{O}(1)$ term is of the form

$$\psi\left(\frac{\log(n) - \log\log(n)}{\log(m)}\right) + o(1)$$

where ψ is a particular periodic function having period 1. Pennington [30] obtained De Bruijn's result by deriving an analogous formula for a general class of partition problems. In chapter 6, we provide an elementary proof of the main term of Mahler's asymptotic formula:

$$\log_m(P_m(n)) \sim \frac{1}{2}(\log_m(n))^2 \tag{1.3}$$

For the sake of comparative study, it is useful to consider ordinary integer partitions $\{p(n)\}_{n=0}^{\infty}$ where $p(n)$ is the number of solutions of

$$n = i_1 + 2i_2 + 3i_3 + \dots$$

Compare 1.3 with the following result due to Hardy and Ramanujan [17]

$$\log(p(n)) \sim \pi\sqrt{\frac{2n}{3}}$$

While somewhat tangential to the topic of m-ary partitions, I believe that it is necessary to mention what motivated us to study them. Our interest in m-ary partitions stems from our interest in the ubiquitous Gamma function ([11] pp. 235-264), which is given by the Weierstrass product

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}}$$

for $z \in \mathbb{C} - \{0, -1, -2, \dots\}$. Here $\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log(n)\right) \approx 0.577216$ is the Euler-Mascheroni constant. We briefly explored available literature on functional equations of the form

$$h(g(z)) = zh(z) \tag{1.4}$$

and the related Schröder's equation:

$$h(g(z)) = ch(z) \tag{1.5}$$

where the function g and the constant c are known; see [10] (German) for more details. Γ is a solution of 1.4 when $g(z) = z + 1$; in fact Γ is the only solution on $x > 0$ if one additionally assumes $h(1) = 1$ and imposes logarithmic convexity on h: this is known as the Bohr-Mollerup theorem; Artin [9] discusses a handful of such results concerning Γ. Consult [14] (French) for Koenig's proof of the existence of a function satisfying Schröder's equation (1.5) when certain conditions are assumed. While most of the results that we proved during our study of m-ary partitions seem to be a part of existing literature, we obtained them independently, without the knowledge of the existence of such literature. With $g(z) = 1 - (1 - z)^m$, making the substitutions $w = 1 - z$ and $f(z) = h(1 - z)$ transforms 1.4 into

$$f(w^m) = (1 - w)f(w)$$

of which, the generating function of $\{P_m(n)\}_{n=0}^{\infty}$ is a solution; we start the next chapter with this result.

Chapter 2

The main recurrence relation

Let

$$D := \{z \mid z \in \mathbb{C},\ |z| < 1\}$$

$$f_m(z) := \prod_{n=0}^{\infty} \frac{1}{1 - z^{m^n}} = \sum_{n=0}^{\infty} P_m(n) z^n \tag{2.1}$$

for $z \in D$. Detailed proofs of the convergence of the above quantities and a proof of the second equality are present in appendix A.

Proposition 1

Consider the following functional equation

$$f(z^m) = (1 - z) f(z)$$

where $f \colon D \to \mathbb{C}$, f is continuous, and $f(0) = 1$. Then $f(z) = f_m(z)$ for $z \in D$.

Proof. With

$$f(z) = \frac{1}{(1 - z)} f(z^m)$$

as the base case, a simple induction argument on $r \geq 0$ gives

$$f(z) = \frac{1}{(1-z)(1-z^m)(1-z^{m^2})\dots(1-z^{m^r})} f(z^{m^{r+1}}) \tag{2.2}$$

In equation 2.2, let $r \to \infty$. By the continuity requirement on f, we have $f(z^{m^{r+1}}) \to f(0) = 1$. Therefore $f(z) = f_m(z)$.

■

$\{P_m(n)\}_{n=0}^{\infty}$ possesses great structure that makes it very easy to study; in proposition 2 and proposition 3, we prove two such simple, yet amazing formulae involving $P_m(n)$; all of these can be seen as being consequences of proposition 1.

Proposition 2

$$P_m(n) = P_m\left(m\left\lfloor \frac{n}{m} \right\rfloor\right) \tag{2.3}$$

Proof. By proposition 1, one has

$$\sum_{i=0}^{\infty} P_m(i) z^{mi} = 1 + \sum_{i=1}^{\infty} (P_m(i) - P_m(i-1)) z^i \tag{2.4}$$

Suppose that n is not a multiple of m; this is same as saying $r := n - m\left\lfloor \frac{n}{m} \right\rfloor > 0$. From the above equation, by comparing coefficients, one gets

$$P_m(n-(k-1)) - P_m(n-k) = 0 \quad \text{for} \;\; 1 \leq k \leq r$$

Thus

$$\begin{aligned} P_m(n) - P_m\left(m\left\lfloor \frac{n}{m} \right\rfloor\right) &= P_m(n) - P_m(n-r) \\ &= \sum_{k=1}^{r} (P_m(n-(k-1)) - P_m(n-k)) \\ &= 0 \end{aligned}$$

Alternatively, one can argue that the only way to represent r as a sum of powers of m is by writing it as a sum of 1s.

$\blacksquare$

The following recurrence relation is the main result of this chapter.

Proposition 3

$$P_m(n) = \sum_{i=0}^{\lfloor \frac{n}{m} \rfloor} P_m(i) \tag{2.5}$$

We present two proofs here.

Proof 1. From 2.4, for $i \geq 1$, we have

$$P_m(mi) - P_m(mi-1) = P_m(i)$$

and

$$P_m(mi-(k-1)) - P_m(mi-k) = 0 \quad \text{for} \;\; 2 \leq k \leq m$$

Add all the above equations to obtain

$$P_m(mi) = P_m(m(i-1)) + P_m(i) \tag{2.6}$$

An induction argument now gives

$$P_m(mi) = \sum_{j=0}^{i} P_m(j)$$

Taking $i = \lfloor \frac{n}{m} \rfloor$ and using proposition 2 now completes the proof.

$\blacksquare$

Proof 2. Consider a particular representation of $m\left\lfloor \frac{n}{m}\right\rfloor$ as a sum of powers of m

$$m\left\lfloor \frac{n}{m}\right\rfloor = j_1 + j_2 m + j_3 m^2 + \dots$$

This implies that $m|j_1$. Thus

$$P_m\left(m\left\lfloor \frac{n}{m}\right\rfloor\right) = \sum_{k=0}^{\lfloor \frac{n}{m}\rfloor} A_m\left(m\left\lfloor \frac{n}{m}\right\rfloor - mk\right)$$

where $A_m(0) := 1$, and for $i \geq 1$, $A_m(i)$ is the number of ways of writing i as a sum of powers of m that are not equal to 1. It is not hard to show that $A_m(mi) = P_m(i)$. By proposition 2, we have

$$P_m(n) = P_m\left(m\left\lfloor \frac{n}{m}\right\rfloor\right) = \sum_{k=0}^{\lfloor \frac{n}{m}\rfloor} P_m\left(\left\lfloor \frac{n}{m}\right\rfloor - k\right) = \sum_{i=0}^{\lfloor \frac{n}{m}\rfloor} P_m(i)$$

■

2.5, 2.3, and 2.6 are the most important formulae in our study, and we will use them extensively, in conjunction with mathematical induction, to obtain simple, yet beautiful results.

Chapter 3

An explicit formula

The goal of this chapter is to iterate 2.5 to obtain an explicit formula for $P_m(n)$; proposition 5 gives us a generalized version of 2.5 which we obtain by iteration. Then, we use proposition 5 to derive the main result, which is stated as proposition 6. At first, we need to recall some classical results.

The Bernoulli polynomials $\{B_n(x)\}_{n=0}^{\infty}$ are given by their exponential generating function

$$\sum_{i=0}^{\infty} \frac{B_i(x)}{i!} t^n := \frac{te^{xt}}{e^t - 1}$$

$$B_i := B_i(0)$$

is rational for all $i \geq 0$; these are known as Bernoulli numbers. In fact

$$B_i(x) = \sum_{k=0}^{i} \binom{i}{k} B_{i-k} x^k$$

Let ζ be the Riemann zeta function. Then

$$\begin{aligned}
B_0 &= 1 \\
B_1 &= -\tfrac{1}{2} \\
B_{2k} &= \tfrac{2(-1)^{k+1}(2k)!\zeta(2k)}{(2\pi)^{2k}} && \text{for } k \geq 1 \\
B_{2k+1} &= 0 && \text{for } k \geq 1
\end{aligned}$$

Let $j \geq 0$.

$$F_j(x) := \frac{B_{j+1}(x+1) - B_{j+1}}{j+1}$$

$\deg(F_j(x)) = j + 1$. Faulhaber's formula says

$$\sum_{i=0}^{n} i^j = F_j(n)$$

assuming $0^0 = 1$. Many other interesting results on Bernoulli polynomials are listed in [25].

Our next goal is to develop tools that are necessary to state and prove proposition 5; consider the following recurrence relation.

Definition

$$g_1(m, n, k) = 1$$

$$g_{N+1}(m, n, k) = \sum_{i=mk}^{\left\lfloor \frac{n}{m^N} \right\rfloor} g_N(m, n, i)$$

for $N \geq 1$.

Proposition 4

$g_N(m, n, k)$ is a polynomial in k of degree $N - 1$.

Proof. The $N = 1$ case is clear. Suppose that the claim holds for $N = K \geq 1$; write

$$g_K(m, n, k) = \sum_{j=0}^{K-1} c_j k^j$$

where c_j does not depend on k. For $j \geq 1$, we have

$$F_j(-1) = \frac{B_{j+1}(0) - B_{j+1}}{j+1} = 0$$

Therefore

$$\begin{aligned} g_{K+1}(m,n,k) &= \sum_{i=mk}^{\left\lfloor \frac{n}{m^K} \right\rfloor} g_K(m,n,i) \\ &= \sum_{j=0}^{K-1} c_j \left(F_j \left(\left\lfloor \frac{n}{m^K} \right\rfloor \right) - F_j(mk-1) \right) \end{aligned}$$

is a polynomial in k of degree K in which the k^K term comes from $F_{K-1}(mk-1)$.

■

By virtue of proposition 4, let us write

$$g_N(m,n,x) = \sum_{j=0}^{N-1} \theta_{N,j}(m,n) x^j$$

for $N \geq 1$. We are at our first milestone:

Proposition 5

$$P_m(n) = \sum_{i=0}^{\left\lfloor \frac{n}{m^N} \right\rfloor} g_N(m,n,i) P_m(i) \tag{3.1}$$

for $n \geq 0$, $N \geq 1$.

Proof. We will use induction for this. The base case is just 2.5. Let us assume that the claim holds for $N = K \geq 1$. Then

$$\begin{aligned} P_m(n) &= \sum_{i=0}^{\left\lfloor \frac{n}{m^K} \right\rfloor} g_K(m,n,i) P_m(i) \\ &= \sum_{i=0}^{\left\lfloor \frac{n}{m^K} \right\rfloor} g_K(m,n,i) \sum_{k=0}^{\left\lfloor \frac{i}{m} \right\rfloor} P_m(k) \\ &= \sum_{i=0}^{\left\lfloor \frac{n}{m^K} \right\rfloor} \sum_{k=0}^{\left\lfloor \frac{i}{m} \right\rfloor} g_K(m,n,i) P_m(k) \\ &= \sum_{k=0}^{\left\lfloor \frac{n}{m^{K+1}} \right\rfloor} \sum_{i=mk}^{\left\lfloor \frac{n}{m^K} \right\rfloor} g_K(m,n,i) P_m(k) \end{aligned}$$

$$= \sum_{k=0}^{\left\lfloor \frac{n}{m^{K+1}} \right\rfloor} g_{K+1}(m,n,k) P_m(k)$$

$\blacksquare$

The first three g_N functions are listed here.

$$\begin{aligned}
g_1(m,n,k) &= 1 \\
g_2(m,n,k) &= \left(\left\lfloor \tfrac{n}{m} \right\rfloor + 1\right) + (-m)k \\
g_3(m,n,k) &= \left(\left(\left\lfloor \tfrac{n}{m} \right\rfloor + 1\right)\left(\left\lfloor \tfrac{n}{m^2} \right\rfloor + 1\right) - \frac{m\left\lfloor \frac{n}{m^2} \right\rfloor\left(\left\lfloor \frac{n}{m^2} \right\rfloor + 1\right)}{2}\right) \\
&\quad + \left(\left(\left\lfloor \tfrac{n}{m} \right\rfloor + 1\right)(-m) - \tfrac{m^2}{2}\right)k + \left(\tfrac{m^3}{2}\right)k^2
\end{aligned}$$

The g_N functions were used to compute the following list of P_m values; Python code that was used to compute the g_N functions is present in appendix B.

$$\begin{aligned}
P_m(k) &= 1 \text{ if } 0 \le k \le m-1 \\
P_m(m) &= 2 \\
P_m(m^2) &= m+2 \\
P_m(m^3) &= \tfrac{m^3}{2} + \tfrac{m^2}{2} + m + 2 \\
P_m(m^4) &= \tfrac{m^6}{6} + \tfrac{m^5}{4} + \tfrac{m^4}{3} + \tfrac{3m^3}{4} + \tfrac{m^2}{2} + m + 2 \\
P_m(m^5) &= \tfrac{m^{10}}{24} + \tfrac{m^9}{12} + \tfrac{m^8}{8} + \tfrac{5m^7}{24} + \tfrac{3m^6}{8} + \tfrac{11m^5}{24} + \tfrac{11m^4}{24} + \tfrac{3m^3}{4} + \tfrac{m^2}{2} + m + 2
\end{aligned}$$

Before moving on to the main result, we need the following definitions.

Definition

For $j \geq 0$ and $i \geq 1$,

$$\beta_{i,j} := \begin{cases} -\frac{1}{j+1}\binom{j+1}{i}B_{j-i+1} & \text{for } 1 \leq i \leq j+1 \\ 0 & \text{for } i \geq j+2 \end{cases}$$

$$A_N(m,n) := \begin{bmatrix} F_0\left(\left\lfloor \frac{n}{m^N} \right\rfloor\right) & F_1\left(\left\lfloor \frac{n}{m^N} \right\rfloor\right) & F_2\left(\left\lfloor \frac{n}{m^N} \right\rfloor\right) & \dots & F_{N-1}\left(\left\lfloor \frac{n}{m^N} \right\rfloor\right) \\ m\beta_{1,0} & m\beta_{1,1} & m\beta_{1,2} & \dots & m\beta_{1,N-1} \\ 0 & m^2\beta_{2,1} & m^2\beta_{2,2} & \dots & m^2\beta_{2,N-1} \\ 0 & 0 & m^3\beta_{3,2} & \dots & m^3\beta_{3,N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & m^N\beta_{N,N-1} \end{bmatrix}_{(N+1)\times N}$$

$$\Theta_N(m,n) := \begin{bmatrix} \theta_{N,0}(m,n) \\ \theta_{N,1}(m,n) \\ \theta_{N,2}(m,n) \\ \vdots \\ \theta_{N,N-1}(m,n) \end{bmatrix}_{N\times 1}$$

We need the next two lemmas to prove proposition 6.

Lemma 1

For $j \geq 0$ and $i \geq 1$,

$$F_j^{(i)}(-1) = -i!\beta_{i,j}$$

Proof.

$$F_j^{(i)}(-1) = \frac{B_{j+1}^{(i)}(0)}{j+1} = \begin{cases} \frac{i!}{j+1}\binom{j+1}{i} B_{j+1-i} & \text{for } 1 \leq i \leq j+1 \\ 0 & \text{for } i \geq j+2 \end{cases}$$

■

The following lemma enables us to write $\theta_{N+1,i}(m,n)$ for $0 \leq i \leq N$ in terms of $\theta_{N,j}(m,n)$ for $0 \leq j \leq N-1$ in the form of a concise matrix equation.

Lemma 2

$$\Theta_{N+1}(m,n) = A_N(m,n)\Theta_N(m,n)$$

for $N \geq 1$.

Proof. Recall that

$$g_{N+1}(m,n,k) = \sum_{j=0}^{N-1} \theta_{N,j}(m,n)\left(F_j\left(\left\lfloor \frac{n}{m^N} \right\rfloor\right) - F_j(mk-1)\right)$$

For $i \geq 1$, we get the following by differentiating with respect to x successively

$$g_{N+1}^{(i)}(m,n,x) = -m^i \sum_{j=0}^{N-1} \theta_{N,j}(m,n) F_j^{(i)}(mx-1)$$

Note that

$$\begin{aligned} \theta_{N+1,0}(m,n) &= g_{N+1}(m,n,0) \\ \theta_{N+1,i}(m,n) &= \tfrac{1}{i!} g_{N+1}^{(i)}(m,n,0) \quad \text{for } 1 \leq i \leq N \end{aligned}$$

Therefore

$$\theta_{N+1,0}(m,n) = \sum_{j=0}^{N-1} F_j\left(\left\lfloor \frac{n}{m^N} \right\rfloor\right) \theta_{N,j}(m,n) \tag{3.2}$$

and

$$\theta_{N+1,i}(m,n) = -\frac{m^i}{i!}\sum_{j=0}^{N-1} F_j^{(i)}(-1)\theta_{N,j}(m,n) \tag{3.3}$$

for $1 \leq i \leq N$. By an application of lemma 1, equation 3.3 becomes

$$\theta_{N+1,i}(m,n) = \sum_{j=0}^{N-1} \left(\beta_{i,j} m^i\right) \theta_{N,j}(m,n) \tag{3.4}$$

for $1 \leq i \leq N$. Finally, write 3.2 and 3.4 in matrix form to obtain the result.

■

We are at the main result of this chapter:

Proposition 6

Let $n \geq m$. Then

$$P_m(n) = B(m,n)A_{L-1}(m,n)A_{L-2}(m,n)\dots A_1(m,n)$$

where $L = \lfloor \log_m(n) \rfloor$, and

$$B(m,n) = \left[\begin{array}{cccc} F_0\left(\left\lfloor \frac{n}{m^L}\right\rfloor\right) & F_1\left(\left\lfloor \frac{n}{m^L}\right\rfloor\right) & \dots & F_{L-1}\left(\left\lfloor \frac{n}{m^L}\right\rfloor\right) \end{array}\right]$$

is the first row of $A_L(m,n)$.

Proof. Noting that $\Theta_1(m,n) = [1]$, lemma 2 and a simple induction argument gives

$$\Theta_N(m,n) = A_{N-1}(m,n)A_{N-2}(m,n)\dots A_1(m,n)$$

for $N \geq 2$. Thus $\theta_{1,0}(m,n) = 1$, and for $N \geq 2$,

$$\theta_{N,0}(m,n) = \Theta_N(m,n)_{1,1} = \left(A_{N-1}(m,n)A_{N-2}(m,n)\dots A_1(m,n)\right)_{1,1}$$

If $n \geq m$, that is, if $\lfloor \log_m(n) \rfloor = L \geq 1$, then $\left\lfloor \frac{n}{m^{L+1}} \right\rfloor = 0$. By proposition 5, we get

$$\begin{aligned} P_m(n) &= \sum_{i=0}^{\left\lfloor \frac{n}{m^{L+1}} \right\rfloor} g_{L+1}(m,n,i) P_m(i) \\ &= g_{L+1}(m,n,0) \\ &= \theta_{L+1,0}(m,n) \\ &= \left(A_L(m,n) A_{L-1}(m,n) \ldots A_1(m,n)\right)_{1,1} \end{aligned}$$

The result easily follows from this.

■

If one has tables of $\beta_{i,j}$ values, then assuming that the F_j polynomials can be calculated efficiently, the formula in proposition 6 will provide a great improvement over simply using 2.5; the matrix multiplications can be performed in time that is polynomial in $\log(n)$.

<u>*Definition*</u>

$$\phi_N(x) := B_N A_{N-1}(x, x^N) A_{N-2}(x, x^N) \ldots A_1(x, x^N)$$

where

$$B_N = \begin{bmatrix} 2 & 1 & 1 & \ldots & 1 \end{bmatrix}_{1 \times N}$$

By proposition 6, we have $P_m(m^N) = \phi_N(m)$. Clearly, $\phi_N(x)$ is a polynomial with rational coefficients. It is not hard to show that the leading term of $\phi_N(x)$ is

$$\frac{1}{(N-1)!} x^{\frac{N(N-1)}{2}}$$

and that

$$\phi_N(0) = 2$$

The following question naturally arises: does m divide $P_m(m^N) - 2 = \phi_N(m) - \phi_N(0)$? The answer is affirmative:

Proposition 7

$$P_m(km^{N+L} - m^L) \equiv 0 \pmod{m}$$

and

$$P_m(km^N) \equiv P_m(km) \pmod{m}$$

for $k \geq 1$, $N \geq 1$, $L \geq 1$. Additionally, if $k \leq m-1$, then

$$P_m(km^N) \equiv k+1 \pmod{m}$$

Conclude that

$$P_m(m^N) \equiv 2 \pmod{m}$$

Proof. Assuming the conditions given in the proposition statement, we have

$$\begin{aligned}
P_m(m(km^N - 1)) &= \sum_{i=0}^{km^N-1} P_m(i) \\
&= \sum_{q=0}^{km^{N-1}-1} \sum_{r=0}^{m-1} P_m(mq+r) \\
&= \sum_{q=0}^{km^{N-1}-1} \sum_{r=0}^{m-1} P_m(mq) \\
&= m \sum_{q=0}^{km^{N-1}-1} P_m(mq)
\end{aligned}$$

$$\equiv 0 \pmod m$$

An induction argument will be used for the second congruence. The $N = 1$ case is clear. Assuming that it holds for $N = K \geq 1$, we have

$$P_m(km^{K+1}) = P_m(m(km^K - 1)) + P_m(km^K) \equiv P_m(km^K) \equiv P_m(km) \pmod m$$

Verifying that $km^N - 1 \geq 1$, one can use the second congruence to get

$$P_m(m^L(km^N - 1)) \equiv P_m(m(km^N - 1)) \equiv 0 \pmod m$$

Finally, we have

$$P_m(km^N) \equiv P_m(km) = \sum_{i=0}^{k} P_m(i) = k + 1 \pmod m$$

■

Chapter 4

A formula involving a restricted sum

The goal of this chapter is to derive a formula for $P_m(n)$ based on proposition 8, which presents the reader with a formula for f_m involving $\Phi_m(x) := 1 + x + x^2 + \cdots + x^{m-1}$.

Proposition 8

$$f_m(x) = \prod_{k=1}^{\infty} \left(\Phi_m \left(x^{m^{k-1}} \right) \right)^k$$

for $x \in \mathbb{R}$, $0 \leq x < 1$.

Proof. Note that

$$\begin{aligned} f_m(x) &= \frac{1}{\prod_{i=0}^{\infty}(1 - x^{m^i})} \\ &= \frac{\Phi_m(x)}{(1 - x^m)^2 \prod_{i=2}^{\infty}(1 - x^{m^i})} \end{aligned}$$

With this as the base case, a simple induction argument on $r \geq 0$ gives

$$f_m(x) = \frac{\prod_{k=1}^{r+1} \left(\Phi_m\left(x^{m^{k-1}}\right)\right)^k}{(1-x^{m^{r+1}})^{r+2} \prod_{i=(r+2)}^{\infty} (1-x^{m^i}) \dots}$$

for $x \in \mathbb{R}$, $0 \leq x < 1$. In the above, if we let $r \to \infty$, then

$$\prod_{i=(r+2)}^{\infty} \left(1 - x^{m^i}\right) \to 1$$

since it is the tail of a convergent infinite product. Also

$$\begin{aligned}
\log\left(\left(1-x^{m^{r+1}}\right)^{r+2}\right) &= (r+2)\log\left(1-x^{m^{r+1}}\right) \\
&= -(r+2)\left(x^{m^{r+1}} + \frac{x^{3m^{r+1}}}{3} + \frac{x^{5m^{r+1}}}{5} + \dots\right) \\
&\to 0
\end{aligned}$$

as $r \to \infty$, since $(r+2)x^{m^{r+1}} \to 0$ as $r \to \infty$. Therefore

$$\left(1 - x^{m^{r+1}}\right)^{r+2} \to 1$$

as $r \to \infty$, and we obtain the desired result as a consequence.

■

Now,

$$\begin{aligned}
(\Phi_m(x))^r &= (1-x^m)^r (1-x)^{-r} \\
&= \left(\sum_{i=0}^{r} \binom{r}{i} (-x^m)^i\right) \left(\sum_{j=0}^{\infty} \binom{r+j-1}{j} x^j\right) \\
&= \sum_{n=0}^{r(m-1)} a(m,n,r) x^n
\end{aligned}$$

where

$$a(m,n,r) = \sum_{\substack{im+j=n \\ 0\le i\le r \\ 0\le j}} (-1)^i \binom{r}{i} \binom{r+j-1}{j}$$

Thus

$$\left(\Phi_m\left(x^{m^{r-1}}\right)\right)^r = \sum_{n=0}^{r(m-1)} a(m,n,r) x^{nm^{r-1}}$$

By proposition 8, we obtain the main result of this chapter:

$$P_m(n) = \sum_{\substack{\sum_{l=1}^{\infty} i_l m^{l-1} = n \\ j(m-1)\ge i_j \ge 0}} \prod_{l=1}^{\infty} a(m, i_l, l)$$

Chapter 5

Analog of Euler's Pentagonal number theorem

Euler's Pentagonal number theorem states that

$$\prod_{i=1}^{\infty}(1-x^i)=\sum_{k=1}^{\infty}(-1)^k\left(x^{\frac{k(3k+1)}{2}}+x^{\frac{k(3k-1)}{2}}\right)$$

This identity can be used to derive the following recurrence relation for $p(n)$ ([25], p. 825):

$$p(n)-p(n-1)-p(n-2)+p(n-5)+p(n-7)-p(n-12)-p(n-15)+\cdots=0$$

where the sum is over generalized pentagonal numbers $\leq n$ and the sign of the k^{th} term is $(-1)^{\left\lfloor\frac{k+1}{2}\right\rfloor}$ for $k\geq 0$. This recurrence relation enabled Major MacMahon to compute $p(n)$ values very efficiently. We will use the same idea here to obtain a similar recurrence relation for P_m. The analog of Euler's Pentagonal theorem in our case is

Proposition 9

$$\prod_{i=0}^{\infty}(1-z^{m^i})=\sum_{(i_1,i_2,\dots)\in\{0,1\}^{\mathbb{N}}}(-1)^{\sum_{k=1}^{\infty}i_k}z^{\sum_{k=1}^{\infty}i_k m^{k-1}}$$

for $z \in D$.

Proof. Consult appendix A for a proof of why the LHS converges for $z \in D$. The series in the RHS converges absolutely since

$$\sum_{(i_1,i_2,\dots,i_N)\in\{0,1\}^N} |z|^{\sum_{k=1}^{N} i_k m^{k-1}} \le \sum_{i=0}^{\infty} |z|^i = \frac{1}{1-|z|}$$

for $N \geq 1$.

We will use induction to show that

$$\prod_{i=0}^{N-1}(1-z^{m^i}) = \sum_{(i_1,i_2,\dots,i_N)\in\{0,1\}^N} (-1)^{\sum_{k=1}^{N} i_k} z^{\sum_{k=1}^{N} i_k m^{k-1}}$$

for $N \geq 1$. The base case is clear. Suppose that the claim holds for $N = K \geq 1$. Then

$$\begin{aligned}\prod_{i=0}^{K}(1-z^{m^i}) &= (1-z^{m^K}) \sum_{(i_1,i_2,\dots,i_K)\in\{0,1\}^K} (-1)^{\sum_{k=1}^{K} i_k} z^{\sum_{k=1}^{K} i_k m^{k-1}} \\ &= \sum_{(i_1,i_2,\dots,i_{K+1})\in\{0,1\}^{K+1}} (-1)^{\sum_{k=1}^{K+1} i_k} z^{\sum_{k=1}^{K+1} i_k m^{k-1}}\end{aligned}$$

Now, let $N \to \infty$.

■

By proposition 9, we have

$$\begin{aligned}1 &= f_m(z)\prod_{i=0}^{\infty}(1-z^{m^i}) \\ &= \left(\sum_{i=0}^{\infty} P_m(i) z^i\right)\left(\sum_{(i_1,i_2,\dots)\in\{0,1\}^{\mathbb{N}}} (-1)^{\sum_{k=1}^{\infty} i_k} z^{\sum_{k=1}^{\infty} i_k m^{k-1}}\right)\end{aligned}$$

Therefore, for $n \geq 1$

$$0 = \sum_{\substack{(i_1,i_2,\dots)\in\{0,1\}^{\mathbb{N}} \\ n \geq \sum_{k=1}^{\infty} i_k m^{k-1}}} (-1)^{\sum_{k=1}^{\infty} i_k} P\left(n - \sum_{k=1}^{\infty} i_k m^{k-1}\right)$$

Thus

$$P_m(n) = \sum_{\substack{(i_1,i_2,\dots)\in\{0,1\}^{\mathbb{N}} \\ n\geq\sum_{k=1}^{\infty} i_k m^{k-1}>0}} (-1)^{1+\sum_{k=1}^{\infty} i_k} P\left(n - \sum_{k=1}^{\infty} i_k m^{k-1}\right) \tag{5.1}$$

In 5.1, the quantities $\sum_{k=1}^{\infty} i_k m^{k-1}$ are all the positive integers not exceeding n that have digits 0 and 1 in base m:

$$1 < m < 1 + m < m^2 < 1 + m^2 < m + m^2 < 1 + m + m^2 < m^3 < \dots$$

Also, the sign of the corresponding term is $+1$ if the number of 1s in the said base m representation is odd; it is -1 if the number of 1s is even. To calculate the value of $P_m\left(m^N\right)$ using 5.1, one requires

$$1 + \binom{N}{1} + \binom{N}{2} + \dots + \binom{N}{N} = 2^N$$

previous P_m values versus the m^{N-1} previous values that one requires if one uses 2.5; thus, if $m \geq 3$, then 5.1 provides us with a more efficient method for calculating P_m values.

Chapter 6

Bounds and limiting behavior

In this chapter, our goal is to derive some basic inequalities involving P_m. We will use these inequalities to answer some questions about the growth rate of P_m that we think are of primary importance. The most important of these inequalities are listed below.

$(P_m(mq))^2 \leq P_m(m^2q)P_m(q)$	for $q \geq 0$	Prop. 10
$\frac{1}{q} \leq \frac{P_m(q)}{P_m(mq)} \leq \frac{1}{\sqrt{1+\frac{m-1}{m}q}}$	for $q \geq m$	Prop. 12
$1 - \frac{1}{\sqrt{1+\frac{m-1}{m}q}} \leq \frac{P_m(m(q-1))}{P_m(mq)} \leq \frac{q-1}{q}$	for $q \geq m$	Prop. 12
$P_m(n) \leq (2m^{\frac{1}{8}})m^{\frac{1}{2}\log_m(n)(\log_m(n)-1)}$	for $n \geq m$	Prop. 14
Given $\epsilon \in \left(0, \frac{1}{2}\right)$, there exists $C_\epsilon > 0$ such that $C_\epsilon T_\epsilon\left(\left\lfloor \frac{n}{m} \right\rfloor\right) \leq P_m(n)$	for $n \geq 1$	Prop. 15

The first inequality will be used to prove the second chain of inequalities, which in turn implies the the third chain of inequalities; the third chain will be used to prove that $\frac{P_m(n+1)}{P_m(n)} \to 1$ as $n \to \infty$ (proposition 13). The last two inequalities will be employed to give a proof of 1.3. A few other questions concerning convexity/concavity of P_m will also be answered in this chapter.

Proposition 10

$$(P_m(mq))^2 \leq P_m(m^2q)P_m(q)$$

for all $q \geq 0$. If $m = 2$, equality holds if and only if $q = 0$ or $q = 1$; if $m \geq 3$, equality holds if and only if $q = 0$.

Proof.

$$\begin{aligned}
(P_m(mq))^2 &= \left(\sum_{i=0}^{q} P_m(i)\right)^2 \\
&= \sum_{i=0}^{q}(P_m(i))^2 + 2\sum_{i=0}^{q-1}\left(\left(\sum_{j=i+1}^{q} P_m(j)\right) P_m(i)\right) \\
&\leq P_m(q)\sum_{i=0}^{q} P_m(i) + 2P_m(q)\sum_{i=0}^{q-1}(q-i)P_m(i) \\
&= P_m(q)\sum_{i=0}^{q}(2(q-i)+1)P_m(i) \\
&\leq P_m(q)\sum_{i=0}^{q}\left(m(q-i)+1\right)P_m(i) \\
&= P_m(q)\sum_{i=0}^{q} g_2(m, m^2q, i)P_m(i) \\
&= P_m(q)P_m(m^2q)
\end{aligned}$$

■

Definition

A sequence $\{b_n\}_{n=0}^{\infty}$ of real numbers is called *concave* if

$$2b_n \geq b_{n-1} + b_{n+1}$$

for all $n \geq 1$. $\{b_n\}_{n=0}^{\infty}$ is called *convex* if

$$2b_n \leq b_{n-1} + b_{n+1}$$

for all $n \geq 1$. A sequence $\{a_n\}_{n=0}^{\infty}$ of nonnegative real numbers is called *log-concave* if

$$a_n^2 \geq a_{n-1}a_{n+1}$$

for all $n \geq 1$. $\{a_n\}_{n=0}^{\infty}$ is called *log-convex* if

$$a_n^2 \leq a_{n-1}a_{n+1}$$

for all $n \geq 1$.

Let $r_N := P_m(m^N)$. Then, by proposition 10, we have that $\{r_n\}_{n=0}^{\infty}$ is log-convex. DeSalvo and Pak proved that $\{p(n)\}_{n=26}^{\infty}$ is log-concave, using Lehmer's estimates on the remainders of the Hardy-Ramanujan and the Rademacher series for $p(n)$ (see [29], [8], [7]). The following proposition answers some basic questions on the concavity/convexity properties of P_m.

Proposition 11

$\{P_m(n)\}_{n=N}^{\infty}$ and $\{P_m(mn)\}_{n=N}^{\infty}$ are neither log-convex nor log-concave for any $N \geq 0$. $\{P_m(n)\}_{n=N}^{\infty}$ is neither convex nor concave for any $N \geq 0$. $\{P_m(mn)\}_{n=0}^{\infty}$ is convex.

Proof. We divide our investigation into four parts.

1. Log-concavity/log-convexity of $\{P_m(n)\}_{n=0}^{\infty}$

Let $q \geq 1$. Then

$$(P_m(mq))^2 = P_m(mq)P_m(mq+1) > P_m(mq-1)P_m(mq+1)$$

whereas

$$(P_m(mq-1))^2 = P_m(mq-2)P_m(mq-1) < P_m(mq-2)P_m(mq)$$

2. Log-concavity/log-convexity of $\{P_m(mn)\}_{n=0}^{\infty}$

$$P_m(m(q+1))P_m(m(q-1)) - (P_m(mq))^2$$

$$= (P_m(mq) + P_m(q+1))(P_m(mq) - P_m(q)) - (P_m(mq))^2$$

$$= P_m(mq)(P_m(q+1) - P_m(q)) - P_m(q+1)P_m(q)$$

Therefore, if $m \not| q+1$, then $P_m(q+1) = P_m(q)$, and

$$P_m(m(q+1))P_m(m(q-1)) - (P_m(mq))^2 = -(P_m(q))^2 < 0$$

However, if $q+1 = mk$ with $k \geq 2$, then, by proposition 10, we have

$$P_m(m(q+1))P_m(m(q-1)) - (P_m(mq))^2$$

$$= P_m(mq)(P_m(q+1) - P_m(q)) - P_m(q+1)P_m(q)$$

$$= P_m(m(mk-1))P_m(k) - P_m(mk)P_m(mk-1)$$

$$= (P_m(m^2k) - P_m(mk))P_m(k) - P_m(mk)P_m(mk-1)$$

$$= P_m(m^2k)P_m(k) - (P_m(mk))^2$$

$$> 0$$

3. Concavity/convexity of $\{P_m(n)\}_{n=0}^{\infty}$

Suppose that $m \nmid n+1$. Then $P_m(n) = P_m(n+1)$ and

$$P_m(n+1) + P_m(n-1) - 2P_m(n) = P_m(n-1) - P_m(n) \leq 0$$

with the inequality being strict when $m \mid n$. However, if $n+1 = mq$, then $P_m(n) = P_m(n-1)$ and

$$P_m(n+1) + P_m(n-1) - 2P_m(n) = P_m(mq) - P_m(m(q-1)) = P_m(q) > 0$$

4. Convexity of $\{P_m(mn)\}_{n=0}^{\infty}$

$$P_m(m(q+1)) + P_m(m(q-1)) - 2P_m(mq) = P_m(q+1) - P_m(q) \geq 0$$

■

We will now turn our attention to the quantity $\frac{P_m(n+1)}{P_m(n)}$; we seek to study the behavior of this as n grows large. Using elementary methods, we were able to derive the following inequalities (proposition 12), which enabled us to show that $\frac{P_m(n+1)}{P_m(n)} \to 1$ as $n \to \infty$ (proposition 13). However, it is evident that much more accurate answers can be obtained using better asymptotic formulae.

Proposition 12

Let $q \geq m$. Then

$$\frac{P_m\left(\left\lfloor \frac{q}{m} \right\rfloor\right)}{P_m(mq)} \leq \frac{1}{\left(1 + \frac{m-1}{m}q\right)}$$

$$\frac{1}{q} \leq \frac{P_m(q)}{P_m(mq)} \leq \frac{1}{\sqrt{1 + \frac{m-1}{m}q}}$$

and

$$1 - \frac{1}{\sqrt{1 + \frac{m-1}{m}q}} \leq \frac{P_m(m(q-1))}{P_m(mq)} \leq \frac{q-1}{q}$$

Conclude that

$$\frac{P_m(q)}{P_m(mq)} \to 0$$

and

$$\frac{P_m(mq)}{P_m(m(q-1))} \to 1$$

as $q \to \infty$.

Proof. Let $q \geq 0$.

$$\begin{aligned}
P_m(mq) &= \sum_{i=0}^{q} P_m(i) \\
&= \sum_{i=0}^{\lfloor \frac{q}{m} \rfloor} P_m(i) + \sum_{i=\lfloor \frac{q}{m} \rfloor + 1}^{q} P_m(i) \\
&= P_m(q) + \sum_{i=\lfloor \frac{q}{m} \rfloor + 1}^{q} P_m(i) \\
&\geq P_m(q) + \left(q - \left\lfloor \frac{q}{m} \right\rfloor\right) P_m\left(\left\lfloor \frac{q}{m} \right\rfloor + 1\right) \\
&\geq \left(1 + \frac{m-1}{m} q\right) P_m\left(\left\lfloor \frac{q}{m} \right\rfloor\right)
\end{aligned}$$

which establishes the first inequality. Now, apply the first inequality and proposition 10 as follows:

$$\begin{aligned}
\frac{P_m(q)}{P_m(mq)} &\leq \frac{1}{(1 + \frac{m-1}{m} q)} \frac{P_m(q)}{P_m\left(\lfloor \frac{q}{m} \rfloor\right)} \\
&\leq \frac{1}{(1 + \frac{m-1}{m} q)} \frac{P_m\left(m^2 \lfloor \frac{q}{m} \rfloor\right)}{P_m(q)} \\
&\leq \frac{1}{(1 + \frac{m-1}{m} q)} \frac{P_m(mq)}{P_m(q)}
\end{aligned}$$

Therefore

$$\left(\frac{P_m(q)}{P_m(mq)}\right)^2 \leq \frac{1}{(1 + \frac{m-1}{m} q)}$$

and one side of the second chain of inequalities follows. For the other side, we need $q \geq m$; we will use induction for this. The $q = m$ case is clear. Let

$$P_m(mk) \leq kP_m(k)$$

for some $k \geq m$. Then

$$P_m(m(k+1)) = P_m(mk) + P_m(k+1) \leq kP_m(k) + P_m(k+1) \leq (k+1)P_m(k+1)$$

We have proved the second chain of inequalities. The third chain of inequalities now easily follows by applying the second chain of inequalities to the following.

$$\frac{P_m(m(q-1))}{P_m(mq)} = \frac{P_m(mq) - P_m(q)}{P_m(mq)} = 1 - \frac{P_m(q)}{P_m(mq)}$$

$\blacksquare$

Proposition 13

$$\frac{P_m(n+1)}{P_m(n)} \to 1$$

as $n \to \infty$.

Proof. If $m \not| n+1$, then $\frac{P_m(n+1)}{P_m(n)} = 1$. Otherwise, let $n+1 = mq$. Then

$$\frac{P_m(n+1)}{P_m(n)} = \frac{P_m(mq)}{P_m(mq-1)} = \frac{P_m(mq)}{P_m(m(q-1))}$$

The result now follows from proposition 12.

$\blacksquare$

Proposition 14 and proposition 15 present the reader with explicit upper and lower bounds for $P_m(n)$ respectively.

Proposition 14

$$P_m(n) \leq 2 \left\lfloor \frac{n}{m} \right\rfloor \left\lfloor \frac{n}{m^2} \right\rfloor \dots \left\lfloor \frac{n}{m^{\lfloor \log_m(n) \rfloor}} \right\rfloor$$

and

$$P_m(n) \leq (2m^{\frac{1}{8}}) m^{\frac{1}{2}\log_m(n)(\log_m(n)-1)}$$

for $n \geq m$.

Proof. Consider the first inequality. At first, we will verify the claim for n such that $m \leq n \leq m^2 - 1$. Note that in this case, $1 \leq \left\lfloor \frac{n}{m} \right\rfloor \leq m - 1$.

$$\begin{aligned} P_m(n) &= \sum_{i=0}^{\lfloor \frac{n}{m} \rfloor} P_m(i) \\ &= \sum_{i=0}^{\lfloor \frac{n}{m} \rfloor} 1 \\ &= \left\lfloor \frac{n}{m} \right\rfloor + 1 \\ &\leq 2 \left\lfloor \frac{n}{m} \right\rfloor \end{aligned}$$

Next, suppose that the claim holds for all n such that $m \leq n \leq k$ where $k \geq m^2 - 1$. We have to show that the claim holds for $n = k + 1$, and then we will be done by induction. Let us divide this into two cases for simplicity.

Case 1

If m does not divide $k+1$, we have that $P_m(k+1) = P_m(k)$. Also $\left\lfloor \frac{k+1}{m^i} \right\rfloor = \left\lfloor \frac{k}{m^i} \right\rfloor$ for all i such that $1 \leq i \leq \lfloor \log_m(k) \rfloor = \lfloor \log_m(k+1) \rfloor$.

Case 2

$k + 1 = mq$. The claim holds for $n = m(q - 1)$ since

$$m^2 \leq k + 1 \implies 2 \leq m \leq q \implies m \leq m(q-1) = (k+1) - m < k$$

Let

$$R := \left\lfloor \frac{k+1}{m^2} \right\rfloor \left\lfloor \frac{k+1}{m^3} \right\rfloor \cdots \left\lfloor \frac{k+1}{m^{\lfloor \log_m(k+1) \rfloor}} \right\rfloor$$

Thus

$$\begin{aligned} P_m(m(q-1)) &\leq 2 \left\lfloor \frac{m(q-1)}{m} \right\rfloor \left\lfloor \frac{m(q-1)}{m^2} \right\rfloor \cdots \left\lfloor \frac{m(q-1)}{m^{\lfloor \log_m(m(q-1)) \rfloor}} \right\rfloor \\ &\leq 2(q-1) \left\lfloor \frac{mq}{m^2} \right\rfloor \left\lfloor \frac{mq}{m^3} \right\rfloor \cdots \left\lfloor \frac{mq}{m^{\lfloor \log_m(mq) \rfloor}} \right\rfloor \\ &= 2(q-1)R \end{aligned}$$

Now, $m \leq q = \frac{k+1}{m} < k$; therefore, the claim holds for $n = q$:

$$\begin{aligned} P_m(q) &\leq 2 \left\lfloor \frac{q}{m} \right\rfloor \left\lfloor \frac{q}{m^2} \right\rfloor \cdots \left\lfloor \frac{q}{m^{\lfloor \log_m(q) \rfloor}} \right\rfloor \\ &= 2R \end{aligned}$$

Combining everything, we get

$$\begin{aligned} P_m(k+1) &= P_m(mq) \\ &= P_m(m(q-1)) + P_m(q) \\ &\leq 2(q-1)R + 2R \\ &= 2qR \\ &= 2 \left\lfloor \frac{k+1}{m} \right\rfloor \left\lfloor \frac{k+1}{m^2} \right\rfloor \cdots \left\lfloor \frac{k+1}{m^{\lfloor \log_m(k+1) \rfloor}} \right\rfloor \end{aligned}$$

which establishes the first inequality. We will use the first inequality to prove the second one. Let $n \geq m$.

$$\begin{aligned} P_m(n) &\leq 2 \left\lfloor \frac{n}{m} \right\rfloor \left\lfloor \frac{n}{m^2} \right\rfloor \cdots \left\lfloor \frac{n}{m^{\lfloor \log_m(n) \rfloor}} \right\rfloor \\ &\leq 2 \left(\frac{n}{m} \right) \left(\frac{n}{m^2} \right) \cdots \left(\frac{n}{m^{\lfloor \log_m(n) \rfloor}} \right) \\ &= 2n^{\lfloor \log_m(n) \rfloor} m^{-\frac{\lfloor \log_m(n) \rfloor (\lfloor \log_m(n) \rfloor + 1)}{2}} \end{aligned}$$

$$\leq (2m^{\frac{1}{8}})m^{\frac{1}{2}\log_m(n)(\log_m(n)-1)}$$

$\blacksquare$

As an application of the upper bounds obtained in proposition 14, we consider the following question: for large n, what proportion of partitions of n are m-ary partitions for $m \geq 1$, with $P_1(n) := 1$? In other words, how does

$$\frac{1 + \sum_{i=2}^{n} P_i(n)}{p(n)}$$

behave as $n \to \infty$? To answer this question, at first, consider

$$h(x) := \frac{1}{4x\sqrt{3}} e^{\pi\sqrt{\frac{2x}{3}}}$$

Hardy and Ramanujan [17] showed that $p(n) \sim h(n)$. Let $n \geq 2$. By proposition 14, we get

$$\begin{aligned}
\frac{1 + \sum_{i=2}^{n} P_i(n)}{p(n)} &\leq \frac{1 + (n-1)P_2(n)}{p(n)} \\
&\leq \frac{1 + 2^{1+\frac{1}{8}}(n-1)2^{\frac{1}{2}\log_2(n)(\log_2(n)-1)}}{p(n)} \\
&= \frac{1 + 2^{1+\frac{1}{8}}(n-1)2^{\frac{1}{2}\log_2(n)(\log_2(n)-1)}}{h(n)} \frac{h(n)}{p(n)} \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Therefore

$$\frac{1 + \sum_{i=2}^{n} P_i(n)}{p(n)} \to 0$$

as $n \to \infty$. In other words, the aforementioned proportion is small for large n.

Lemma 3

Let $\epsilon \in \left(0, \frac{1}{2}\right)$.

$$T_\epsilon(x) := m^{\epsilon(\log_m(mx))^2}$$

Then

$$T_\epsilon\left(\left\lfloor\frac{q}{m}\right\rfloor\right) - (T_\epsilon(q) - T_\epsilon(q-1)) \to \infty$$

as $q \to \infty$. From this, immediately conclude that there exists $N_\epsilon \geq 1$ such that $q \geq N_\epsilon$ implies

$$T_\epsilon(q) \leq T_\epsilon(q-1) + T_\epsilon\left(\left\lfloor\frac{q}{m}\right\rfloor\right)$$

Proof. Fix $M \geq 1$.

$$T'_\epsilon(x) = \frac{2\epsilon\log_m(mx)}{x} m^{\epsilon(\log_m(mx))^2} = 2\epsilon m^\epsilon \log_m(mx) x^{\epsilon\log_m(m^2x)-1}$$

It is clear that T'_ϵ is an increasing function for $x \geq 1$; therefore, $q \geq 2$ implies $T'_\epsilon(y) \leq T'_\epsilon(q)$ for all $y \in (q-1, q)$; as a consequence, we have

$$T_\epsilon(q) - T_\epsilon(q-1) \leq T'_\epsilon(q)$$

by the mean value theorem. Since M was chosen arbitrarily, it is now sufficient to find $N \geq 2$ such that $q \geq N$ implies

$$M + T'_\epsilon(q) \leq T_\epsilon\left(\left\lfloor\frac{q}{m}\right\rfloor\right)$$

Writing $t = \log_m(q)$ and $\delta = \log_m\left(\frac{\frac{q}{m}}{\left\lfloor\frac{q}{m}\right\rfloor}\right) > 0$, this can be rewritten as

$$M + \frac{2\epsilon(t+1)}{m^t} m^{\epsilon(t+1)^2} \leq m^{\epsilon(t-\delta)^2}$$

or

$$Mm^{-(\epsilon t^2 - t(1-2\epsilon)+\epsilon\delta^2)} + 2\epsilon m^{\epsilon(1-\delta^2)}(t+1) \leq m^{(1-2\epsilon(1+\delta))t}$$

$\delta \to 0$ as $q \to \infty$. Thus, there exists $L \geq 2$ such that $q \geq L$ implies $\delta < \frac{1-2\epsilon}{4\epsilon}$, $Mm^{-(\epsilon t^2 - t(1-2\epsilon)+\epsilon\delta^2)} < 1$. This implies $\frac{1}{2} - \epsilon < 1 - 2\epsilon(1+\delta)$. Therefore, now it is sufficient to find $N \geq L$ such that $q \geq N$

implies

$$1 + 2\epsilon m^{\epsilon}(t+1) \leq m^{\left(\frac{1}{2}-\epsilon\right)t}$$

or

$$1 + 2\epsilon m^{\epsilon}(\log_m(q) + 1) \leq q^{\left(\frac{1}{2}-\epsilon\right)}$$

which is clear.

$\blacksquare$

Proposition 15

Let $\epsilon \in \left(0, \frac{1}{2}\right)$. There exists $C_\epsilon > 0$, which does not depend on n, such that

$$C_\epsilon T_\epsilon \left(\left\lfloor \frac{n}{m} \right\rfloor \right) \leq P_m(n)$$

for $n \geq 1$.

Proof. Choose C_ϵ such that

$$0 < C_\epsilon \leq \frac{P_m(n)}{T_\epsilon\left(\left\lfloor \frac{n}{m} \right\rfloor\right)}$$

for all n such that $1 \leq n \leq mN_\epsilon$. Now, suppose that the claim holds for all n such that $1 \leq n \leq K$ where $K \geq mN_\epsilon$. We will use induction for this. We will divide the argument into two cases for simplicity.

<u>Case 1</u>

If $K+1$ is not divisible by m, then $\left\lfloor \frac{K+1}{m} \right\rfloor = \left\lfloor \frac{K}{m} \right\rfloor$.

<u>Case 2</u>

$K + 1 = mq$. Since $m(q-1) \leq K$ and $q \leq K$, we have

$$\begin{aligned} P_m(K+1) &= P_m(mq) \\ &= P_m(m(q-1)) + P_m(q) \\ &\geq C_\epsilon \left(T_\epsilon(q-1) + T_\epsilon \left(\left\lfloor \frac{q}{m} \right\rfloor \right) \right) \end{aligned}$$

$$
\begin{aligned}
&\geq C_\epsilon T_\epsilon(q) \\
&= C_\epsilon T_\epsilon\left(\left\lfloor \frac{K+1}{m} \right\rfloor\right)
\end{aligned}
$$

by lemma 3 and the fact that $q = \frac{K+1}{m} > N_\epsilon$.

■

Finally, we give a proof of 1.3 as promised. Fix $\epsilon \in \left(0, \frac{1}{2}\right)$. From proposition 14 and proposition 15, one gets

$$
\begin{aligned}
\log(C_\epsilon) + \epsilon\left(\log_m\left(m\left\lfloor \frac{n}{m} \right\rfloor\right)\right)^2 &\leq \log_m(P_m(n)) \\
&\leq \log_m(2m^{\frac{1}{8}}) + \frac{1}{2}\log_m(n)(\log_m(n) - 1)
\end{aligned}
$$

for $n \geq m$. Dividing through by $(\log_m(n))^2$, and letting $n \to \infty$, we get

$$
\epsilon \leq \liminf_{n\to\infty} \frac{\log_m(P_m(n))}{(\log_m(n))^2} \leq \limsup_{n\to\infty} \frac{\log_m(P_m(n))}{(\log_m(n))^2} \leq \frac{1}{2}
$$

Let $\epsilon \to \frac{1}{2}$ to obtain 1.3, which is restated here for convenience of the reader:

$$
\log_m(P_m(n)) \sim \frac{1}{2}(\log_m(n))^2
$$

Appendices

Appendix A Some convergence proofs

1. Proof of the convergence of the infinite product in 2.1

For $z \in D$,

$$\sum_{n=0}^{N} \left| -z^{m^n} \right| = \sum_{n=0}^{N} |z|^{m^n} < \sum_{n=0}^{m^N} |z|^n < \frac{1}{1-|z|}$$

This tells us that

$$\left\{ \sum_{n=0}^{N} \left| -z^{m^n} \right| \right\}_{N=0}^{\infty}$$

is a bounded, monotonically increasing sequence; therefore, it converges. We will now need the following classical result.

If $\sum_{n=0}^{\infty} |a_n| \in \mathbb{R}$, then $\prod_{n=0}^{\infty}(1+a_n) \in \mathbb{C}$. Moreover, the product converges to 0 if and only if one of its factors is 0.

Therefore $\prod_{n=0}^{\infty}(1-z^{m^n}) \in \mathbb{C} - \{0\}$.

■

2. Proof of the convergence of the infinite series in 2.1

<u>*Definition*</u>

$P_{m,N}(0) := 1$; for $n \geq 1$, let $P_{m,N}(n)$ be the number of ways of writing n as a sum of powers of m which do not exceed m^N.

Note that $P_m(n) \geq P_{m,N}(n)$. In particular, we have

$$P_{m,N}(n) = P_m(n)$$

for $0 \leq n \leq m^N$. Let

$$f_{m,N}(z) := \prod_{n=0}^{N} \frac{1}{1-z^{m^n}}$$

This is simply a partial product of the product in equation 2.1.

$$\begin{aligned} f_{m,N}(z) &= \prod_{n=0}^{N} \sum_{j=0}^{\infty} z^{jm^n} \\ &= \sum_{n=0}^{\infty} P_{m,N}(n) z^n \end{aligned}$$

Therefore $f_{m,N}$ is the generating function of $\{P_{m,N}(n)\}_{n=0}^{\infty}$. Now, let

$$S_M(z) := \sum_{n=0}^{M} P_m(n) z^n$$

Then

$$\begin{aligned} S_M(|z|) &\leq \sum_{n=0}^{m^K} P_m(n)|z|^n \quad \text{where } K = \lfloor \log_m(M) \rfloor + 1 \\ &= \sum_{n=0}^{m^K} P_{m,K}(n)|z|^n \\ &\leq \sum_{n=0}^{\infty} P_{m,K}(n)|z|^n \\ &= f_{m,K}(|z|) \\ &\leq f_m(|z|) \end{aligned}$$

The last step works because each term in the infinite product in equation 2.1 is greater than or equal to 1. Therefore $\{S_M(|z|)\}_{M=0}^{\infty}$ is a bounded and monotonically increasing sequence; the conclusion is that

$$\sum_{n=0}^{\infty} P_m(n) z^n$$

is an absolutely convergent series.

■

3. Proof of the second equality in 2.1

$$\sum_{n=0}^{\infty} P_{m,N}(n) z^n = \sum_{n=0}^{m^N} P_m(n) z^n + \sum_{n=m^N+1}^{\infty} P_{m,N}(n) z^n \tag{1}$$

Now

$$\begin{aligned}
\left| \sum_{n=m^N+1}^{\infty} P_{m,N}(n) z^n \right| &\leq \sum_{n=m^N+1}^{\infty} P_{m,N}(n) |z|^n \\
&\leq \sum_{n=m^N+1}^{\infty} P_m(n) |z|^n \\
&\to 0
\end{aligned}$$

as $N \to \infty$ since the penultimate quantity is the tail of a convergent series. Let $N \to \infty$ in equation 1:

$$\begin{aligned}
f_m(z) &= \lim_{N \to \infty} f_{m,N}(z) \\
&= \lim_{N \to \infty} \sum_{n=0}^{\infty} P_{m,N}(n) z^n \\
&= \sum_{n=0}^{\infty} P_m(n) z^n
\end{aligned}$$

Thus f_m is the generating function of $\{P_m(n)\}_{n=0}^{\infty}$; it is analytic in D and

$$P_m(n) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z^{n+1}} \, dz$$

where γ is any path enclosing the origin and lying completely inside the unit circle.

$\blacksquare$

Appendix B Code

```
import sys
from sympy import *

sys.setrecursionlimit(10000)

x = symbols('x')

# p: current polynomial
# m: radix
# next_polynomial(x) = \sum_{i = mx}^{q} current_polynomial(i)
def get_next_polynomial(m, current_polynomial, q):

       i = symbols('i')

       return(Sum(current_polynomial.subs(x, i), (i, m*x, q)).doit())

M = symbols('m')

f = [Integer(1)]

N = 7 # control variable

for t in range(1, N-1):

    Q = symbols('q_{0}'.format(t))
    f.append(get_next_polynomial(M, f[-1], Q))
```

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